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(Conformal) Killing Vectors and their Associated Bivectors

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Abstract

Fayos and Sopuerta have recently set up a formalism for studying vacuum spacetimes with an isometry, a formalism that is centred around the bivector corresponding to the Killing vector and that adapts the tetrad to the bivector. Steele has generalized their approach to include the homothetic case. Here, we generalize this formalism to arbitrary spacetimes and to homothetic and conformal Killing vectors but do not insist on aligning the tetrad with the bivector. The most efficient way to use the formalism to find conformal Killing vectors (proper or not) of a given spacetime is to combine it with the notion of a preferred tetrad. A metric by Kimura is used as an illustrative example.

I. INTRODUCTION

In a number of papers[1,2,3] Brian Edgar and the present author investigated spacetimes with (conformal) Killing vectors ((C)KVs), i.e. with Killing vectors (KVs), a homothetic vector (HV), and/or proper conformal Killing vectors (CKVs). One of the key ingredients was the notion of a *preferred tetrad* relative to a (C)KV. Working, at first, in the Geroch-Held-Penrose (GHP) formalism[4] the notion of *preferred null directions* relative to a vector, in particular, relative to a (C)KV, was defined. This was done in such a way that when a suitably defined GHP generalization of the ordinary Lie derivative is applied to such a preferred GHP tetrad the result takes its simplest possible form. Proceeding to the Newman-Penrose formalism[5] it was then necessary to define the notion of a *preferred gauge* as well (thus defining what is meant by a preferred tetrad). Although this was done by requiring that the the GHP Lie operator and the ordinary Lie derivative have the same effect on arbitrary scalar quantities, the upshot of it was that when the (ordinary) Lie derivative is applied to a preferred tetrad it yields the simplest possible result. In particular, relative to a KV, the Lie derivative annihilates the tetrad if and only if the latter is preferred.

In a recent article, Fayos and Sopuerta[6] (FS) set up a formalism to facilitate the study of vacuum spacetimes with an isometry. (Steele[7] has meanwhile extended their method to include homotheties.) This formalism centred around the bivector associated with a Killing vector. The aim of the present paper is three-fold. Firstly, we show that their formalism can be obtained quite simply by re-writing the Killing equations and their integrability conditions as obtained in Ref.[8] (KL), in terms of the associated bivector. Secondly, since the latter equations were obtained for an arbitrary spacetime, vacuum or not, and for homothetic and proper conformal Killing vectors as well, the extension of the FS equations to this most general case is straightforward. Thirdly, we show how the generalized FS equations or, equivalently, the KL equations, can be used most efficiently to find all (C)KVs for a given spacetime if they are combined with the notion of a preferred tetrad.[2,3]

Normally, to find all possible (C)KVs of a given spacetime one solves the (conformal) Killing equations. Inevitably, this has to be done with the aid of the integrability conditions of these equations. Both sets of equations were worked out in all generality in KL[8] in the GHP formalism. They are readily converted into the NP formalism. Generally speaking, these equations are still quite difficult to tackle unless one chooses the tetrad appropriately. Insisting that the tetrad be preferred relative to a (C)KV yet to be found, furnishes tremendous simplification. In the FS approach the tetrad direction(s) are chosen to be principal null direction(s) of the bivector associated with the KV. In the case of an HV or a KV, such null directions are then preferred and lead to suitable simplifications. (The issue of alignment of the bivector with the Weyl tensor is not addressed here. It applies only to some specialized cases, albeit perhaps interesting ones. Here we concentrate on the more general case where such an alignment may or may not exist.) However, it is usually better to adapt the null directions to the Weyl tensor or some other aspect of the (conformal) geometry since they will then be preferred with respect to *all* (C)KVs. Except when there is alignment, such null directions are then not principal null directions of the bivector(s) associated with the (C)KVs. Further, when dealing with a proper CKV, the principal null directions of the associated bivector are not preferred and the equations will not simplify. Therefore, although the FS formalism (and, by implication, the present extension) appears to be useful in deriving general properties for spacetimes with (C)KVs it does not seem to be a good tool for actually determining such (C)KVs unless combined with the notion of a preferred tetrad. Although we maintain that the best tool is the commutator approach[1-3] (which also employs preferred tetrads), in this paper we shall work with the KL equations or, equivalently, the (generalized) FS equations.

The notation used here for the tetrad components of the (C)KV and of other quantities agrees with that of Refs.[[2,3]] and is different in some respects from that used by FS.[6] However, the correspondence is readily made.

In section 2 we review the notion of preferred null directions relative to a (C)KV and rewrite the conformal Killing equations and their integrabilty conditions in terms of the bivector associated with a (C)KV. In the following section we make the connection to the FS formalism and discuss a few general results that may readily be obtained from this formalism. In section 4, after reviewing the notion of a preferred gauge, we convert these equations from the GHP formalism to the NP formalism. Finally, in section 5, we illustrate on a concrete example (the non-vacuum metric of Kimura[9]) how these equations, when used in conjunction with a preferred tetrad, can be solved to yield all (C)KVs of the given metric.

II. THE GHP CONFORMAL KILLING EQUATIONS AND THEIR INTEGRABILITY CONDITIONS

The most useful generalization of the Lie derivative $\pounds_\xi$ to the GHP formalism is the "GHP Lie derivative $Ł_\xi$" defined as

$$Ł_\xi = L_\xi - \frac{p}{4}\left(\mathcal{P} - \mathcal{P}' + \mathcal{P}^* - \mathcal{P}'^*\right) - \frac{q}{4}\left(\mathcal{P} - \mathcal{P}' + \mathcal{P}'^* - \mathcal{P}^*\right) \tag{1}$$

where p, q are the GHP weights of the quantity operated upon,

$$\mathcal{P} = n_\mu L_\xi l^\mu \tag{2}$$

(with similar definitions for the companions[8] $\mathcal{P}', \mathcal{P}^*, \mathcal{P}'^*$), and

$$L_\xi = \pounds_\xi - \xi^\mu\left(p\zeta_\mu + q\overline{\zeta}_\mu\right), \tag{3}$$

where, using the usual NP notation,

$$\zeta_\mu = \gamma l_\mu + \varepsilon n_\mu - \alpha m_\mu - \beta \overline{m}_\mu . \tag{4}$$

As a result, $Ł_\xi l^\mu$ has the form

$$Ł_\xi l^\mu = \frac{1}{2}\left(\mathcal{P} + \mathcal{P}'\right) l^\mu + \mathcal{R} n^\mu - \mathcal{Q} m^\mu - \overline{\mathcal{Q}}\overline{m}^\mu \tag{5}$$

and similarly for its companions under the prime, star and star-prime operations. When ξ is a (C)KV, i.e. when it satisfies

$$\pounds_\xi g_{\mu\nu} = \varphi g_{\mu\nu}, \tag{6}$$

the conformal Killing equations may be written

$$\begin{aligned} &\mathcal{P}' = -\mathcal{P} - \varphi, \qquad \mathcal{P}'^* = -\mathcal{P}^* - \varphi \\ &\mathcal{Q} = \mathcal{Q}'^*, \qquad \mathcal{Q}' = \mathcal{Q}^* \\ &\mathcal{R} = \mathcal{R}' = \mathcal{R}^* = \mathcal{R}'^* = 0. \end{aligned} \tag{7}$$

Since[10] $\overline{\mathcal{P}^*} = \mathcal{P}'^*$ it follows from the second of Eqs.(7) that $\overline{\mathcal{P}^*} + \mathcal{P}^* = -\varphi$. Hence, for a (C)KV, we can write

$$\mathcal{P}^* = -\frac{\varphi}{2} - i\mathcal{S} \tag{8}$$

where S is real. It follows that for a (C)KV,

$$Ł_\xi = L_\xi - \frac{p}{2}\left(\mathcal{P} - i\mathcal{S} + \frac{\varphi}{2}\right) - \frac{q}{2}\left(\mathcal{P} + i\mathcal{S} + \frac{\varphi}{2}\right). \tag{9}$$

When the null directions l and n are chosen such that $\mathcal{Q} = \mathcal{Q}' = 0$, which is always possible, they are called *preferred*. In this case we have (for a (C)KV ξ)

$$Ł_\xi l^\mu = -\frac{1}{2}\varphi l^\mu, \tag{10}$$

and similarly for its companions.

The bivector $F_{\mu\nu}$ associated with a (C)KV is, as in Eq.(KL24) of Ref.[8], defined by

$$F_{\mu\nu} = \xi_{\mu;\nu} - \frac{1}{2}\varphi g_{\mu\nu} \tag{11}$$

and its tetrad components, as in Eqs.(KL26-KL28), are given by

$$\phi_0 = F_{\mu\nu} l^\mu m^\nu = \overline{\mathcal{Q}} - \kappa a - \tau b + \sigma c + \rho \bar{c} \tag{12}$$

$$\phi_2 = F_{\mu\nu} \overline{m}^\mu n^\nu = -\overline{\mathcal{Q}}' - \pi a - \nu b + \mu c + \lambda \bar{c} \tag{12$'$}$$

$$\phi_1 = \frac{1}{2} F_{\mu\nu}\left(l^\mu n^\nu + \overline{m}^\mu m^\nu\right) = \frac{1}{2}\left(\mathcal{P} - i\mathcal{S} + \frac{\varphi}{2}\right), \tag{13}$$

where a, b, c, d are the tetrad components of the (C)KV ξ:

$$\xi = a l^\mu + b n^\mu - c m^\mu - \overline{c m}^\mu. \tag{14}$$

Note that by the last of Eqs.(12), Eq.(9) may now be written as

$$Ł_\xi = L_\xi - \frac{p}{2}\phi_1 - \frac{q}{2}\overline{\phi_1}. \tag{15}$$

It is also worthwhile noting that the weights of ϕ_0, ϕ_1, ϕ_2 are, respectively, (2,0), (0,0), and (-2,0) and that under the prime and star operations[8,10] these quantities transform according to

$$\begin{aligned} &\phi_0' = -\phi_2, \qquad \phi_1' = -\phi_1' \\ &\phi_i^* = \phi_i \quad (i = 0, 1, 2). \end{aligned} \tag{16}$$

The weights and transformation properties of other GHP quantities of interest are found in the Appendices of Ref.[8].

In terms of these quantities ϕ_i the Killing equations as given by Eqs.(KL21-KL23)

become

$$\text{þ}b = -\kappa c - \overline{\kappa c} \tag{17}$$

$$\text{þ}'a = \overline{\nu}c + \nu\overline{c} \tag{17$'$}$$

$$\eth\overline{c} = -\sigma a + \overline{\lambda}b \tag{17*}$$

$$\text{þ}a = \frac{\varphi}{2} - \phi_1 - \overline{\phi_1} + \overline{\pi}c + \pi\overline{c} \tag{18}$$

$$\text{þ}'b = \frac{\varphi}{2} + \phi_1 + \overline{\phi_1} - \tau c - \overline{\tau c} \tag{18$'$}$$

$$\eth c = -\frac{\varphi}{2} + \phi_1 - \overline{\phi_1} - \overline{\rho}a + \mu b \tag{18*}$$

$$\text{þ}c = -\overline{\phi_0} - \overline{\kappa}a + \pi b \tag{19}$$

$$\text{þ}'\overline{c} = \overline{\phi_2} - \tau a + \overline{\nu}b \tag{19$'$}$$

$$\eth a = -\overline{\phi_2} + \overline{\lambda}c + \mu\overline{c} \tag{19*}$$

$$\overline{\eth}b = \overline{\phi_0} - \rho c - \overline{\sigma c} \tag{19$'$*}$$

Their first integrability conditions, given in Eqs.(KL34-KL36) are also easily re-written in terms of the ϕ_i and become

$$\text{þ}\phi_1 = \pi\phi_0 - \kappa\phi_2 - \frac{1}{4}\text{þ}\varphi + b\left(\Lambda - \Phi_{11} - \Psi_2\right) + c\Psi_1 + \overline{c}\Phi_{10} \tag{20}$$

$$\text{þ}'\phi_1 = \nu\phi_0 - \tau\phi_2 + \frac{1}{4}\text{þ}'\varphi - a\left(\Lambda - \Phi_{11} - \Psi_2\right) - c\Phi_{12} - \overline{c}\Psi_3 \tag{20$'$}$$

$$\eth\phi_1 = \mu\phi_0 - \sigma\phi_2 - \frac{1}{4}\eth\varphi + a\Psi_1 - b\Phi_{12} + \overline{c}\left(\Lambda + \Phi_{11} - \Psi_2\right) \tag{20*}$$

$$\overline{\eth}\phi_1 = \lambda\phi_0 - \rho\phi_2 + \frac{1}{4}\overline{\eth}\varphi + a\Phi_{10} - b\Psi_3 - c\left(\Lambda + \Phi_{11} - \Psi_2\right) \tag{20$'$*}$$

$$\text{þ}\phi_2 = 2\pi\phi_1 - \frac{1}{2}\overline{\eth}\varphi - b\left(\Psi_3 + \Phi_{21}\right) + c\left(\Psi_2 + 2\Lambda\right) + \overline{c}\Phi_{20} \tag{21}$$

$$\text{þ}'\phi_0 = -2\tau\phi_1 + \frac{1}{2}\eth\varphi + a\left(\Psi_1 + \Phi_{01}\right) - c\Phi_{02} - \overline{c}\left(\Psi_2 + 2\Lambda\right) \tag{21$'$}$$

$$\eth\phi_2 = 2\mu\phi_1 - \frac{1}{2}\text{þ}'\varphi + a\left(\Psi_2 + 2\Lambda\right) - b\Phi_{22} + \overline{c}\left(\Phi_{21} - \Psi_3\right) \tag{21*}$$

$$\overline{\eth}\phi_0 = -2\rho\phi_1 + \frac{1}{2}\text{þ}\varphi + a\Phi_{00} - b\left(\Psi_2 + 2\Lambda\right) + c\left(\Psi_1 - \Phi_{01}\right) \tag{21$'$*}$$

$$\text{þ}\phi_0 = -2\kappa\phi_1 - b\left(\Psi_1 + \Phi_{01}\right) + c\Psi_0 + \overline{c}\Phi_{00} \tag{22}$$

$$\text{þ}'\phi_2 = 2\nu\phi_1 + a\left(\Psi_3 + \Phi_{21}\right) - c\Phi_{22} - \overline{c}\Psi_4 \tag{22$'$}$$

$$\eth\phi_0 = -2\sigma\phi_1 + a\Psi_0 - b\Phi_{02} - \overline{c}\left(\Psi_1 - \Phi_{01}\right) \tag{22*}$$

$$\overline{\eth}\phi_2 = 2\lambda\phi_1 + a\Phi_{20} - b\Psi_4 + c\left(\Psi_3 - \Phi_{21}\right) \tag{22$'$*}$$

The Maxwell equations (with source) are implicit in these equations. They are obtained by subtracting Eq.(21$'$*) from Eq.(20) and doing the same with their respective companions.

Thus,

$$\text{þ}\phi_1 - \bar{\eth}\phi_0 = \pi\phi_0 + 2\rho\phi_1 - \kappa\phi_2 - \frac{3}{4}\text{þ}\varphi - a\Phi_{00} + b\,(3\Lambda - \Phi_{11}) + c\Phi_{01} + \bar{c}\Phi_{10} \tag{23}$$

$$\text{þ}'\phi_1 - \eth\phi_2 = \nu\phi_0 - 2\mu\phi_1 - \tau\phi_2 + \frac{3}{4}\text{þ}'\varphi - a\,(3\Lambda - \Phi_{11}) + b\Phi_{22} - c\Phi_{12} - \bar{c}\Phi_{21} \tag{23$'$}$$

$$\eth\phi_1 - \text{þ}'\phi_0 = \mu\phi_0 + 2\tau\phi_1 - \sigma\phi_2 - \frac{3}{4}\eth\varphi - a\Phi_{01} - b\Phi_{12} + c\Phi_{02} + \bar{c}\,(3\Lambda + \Phi_{11}) \tag{23*}$$

$$\bar{\eth}\phi_1 - \text{þ}\phi_2 = \lambda\phi_0 - 2\pi\phi_1 - \rho\phi_2 + \frac{3}{4}\bar{\eth}\varphi + a\Phi_{10} + b\Phi_{21} - \bar{c}\Phi_{20} - c\,(3\Lambda + \Phi_{11}) \tag{23$'$*}$$

When specialized to vacuum and to a proper Killing vector, Eqs.(20) - (22) readily yield the formalism of FS, as we shall see in the next section.

III. GENERAL CONSIDERATIONS

Before revisiting the FS equations, albeit in their most general from which includes non-vacuum metrics and HV and proper CKV, let us derive some general results of interest.

Multiplying Eqs.(22), (21$'$), (22*), and (21$'$*) by $a, b, -c, -\bar{c}$, respectively, and adding the results we first calculate $\left(a\text{þ} + b\text{þ}' - c\eth - \bar{c}\bar{\eth}\right)\phi_0$, i.e. $L_\xi\phi_0$, and then, from Eq.(15), $Ł_\xi\phi_0$. Doing similar calculations for ϕ_1 and ϕ_2 we find that

$$Ł_\xi\phi_0 = -2\overline{\mathcal{Q}}\phi_1 + \frac{1}{2}b\eth\varphi - \frac{1}{2}\bar{c}\text{þ}\varphi \tag{24}$$

$$Ł_\xi\phi_2 = -2\overline{\mathcal{Q}}'\phi_1 - \frac{1}{2}a\bar{\eth}\varphi + \frac{1}{2}c\text{þ}'\varphi \tag{24$'$}$$

$$Ł_\xi\phi_1 = -\overline{\mathcal{Q}}'\phi_0 - \overline{\mathcal{Q}}\phi_2 + \frac{1}{4}\left(-a\text{þ}\varphi + b\text{þ}'\varphi + c\eth\varphi - \bar{c}\bar{\eth}\varphi\right). \tag{25}$$

Restricting ourselves to HV and KVs and assuming that the bivector $F_{\mu\nu}$ does not vanish identically, we see immediately from Eqs.(24, 24$'$, 25) that if the null directions are preferred then the ϕ_i are annihilated by the GHP operator $Ł_\xi$, as perhaps expected. Conversely, if we choose the l - direction to be a principal null direction of the bivector, so that $\phi_0$vanishes, we find immediately from Eqs.(24) and (25) that $\mathcal{Q} = 0$, i.e. that this direction is preferred. From the first of Eqs.(12) we now deduce that

$$\kappa a + \tau b - \sigma c - \rho\bar{c} = 0 \tag{26}$$

i.e. that the vector $X_1 = -\tau l - \kappa n + \rho m + \sigma\overline{m}$ is orthogonal to the KV/HV ξ. If the bivector is non-null we can choose the second null direction to coincide with the bivector's second principal null direction and we get the prime of the above result, $\mathcal{Q}' = 0$, i.e.

$$-\pi a - b\nu + \mu c + \lambda\bar{c} = 0 \tag{27}$$

i.e. that the vector $X_3 = \nu l + \pi n - \lambda m - \mu\overline{m}$ is also orthogonal to the KV/HV ξ. As FS have shown, and as we shall see below, there may, under certain circumstances (such as in vacuum), be yet another vector X_2 orthogonal to ξ.

Still restricting ourselves to an HV or a KV, if we add Eqs.(21*) and (21$'$*), convert to the NP formalism where we take a gauge such that $\rho = \mu$ (which is possible provided $\rho\mu \neq 0$) we obtain after a lengthy calculation that $\rho,_\mu \xi^\mu = \frac{\varphi}{2}\rho$. Clearly now, if ρ is a constant in this gauge, φ has to vanish; there cannot be an HV. This is the case for the Kimura metric considered in Section 5. It should be noted that this conclusion is arrived at much faster using the KL formalism.[8] In fact, Eqs.(21*) and (21$'$*) are simply Eqs.(KL35*) and (KL35$'$*) which, for preferred null directions reduce to the complex

conjugates of $\text{Ł}_\xi \rho = \frac{\varphi}{2}\rho$ and $\text{Ł}_\xi \mu = \frac{\varphi}{2}\mu$, respectively. Since the gauge $\rho = \mu$ has a preferred boost part and since both ρ and μ have weights of the form (p,p), it follows from the discussion in the next section that $\mathcal{L}_\xi \rho = \frac{\varphi}{2}\rho$ and $\mathcal{L}_\xi \mu = \frac{\varphi}{2}\mu$. Hence, when $\rho = \mu =$ constant we necessarily have $\varphi = 0$.

Let us now return to the general case that includes proper CKVs. Eqs.(21)-(22), including their companions, can be solved pairwise for $N\Psi_i$ $(i = 0, ...4))$, where $N = \xi^\mu \xi_\mu$. We find that

$$
\begin{aligned}
N\Psi_3 &= c\left(\eth\phi_2 - 2\mu\phi_1 + \frac{1}{2}\text{þ}'\varphi + b\Phi_{22} - \bar{c}\Phi_{21}\right) \\
&\quad - a\left(\text{þ}\phi_2 - 2\pi\phi_1 + \frac{1}{2}\bar{\eth}\varphi + b\Phi_{21} - \bar{c}\Phi_{20}\right) \qquad (28) \\
N(\Psi_2 + 2\Lambda) &= b\left(\eth\phi_2 - 2\mu\phi_1 + \frac{1}{2}\text{þ}'\varphi + b\Phi_{22} - \bar{c}\Phi_{21}\right) \\
&\quad - \bar{c}\left(\text{þ}\phi_2 - 2\pi\phi_1 + \frac{1}{2}\bar{\eth}\varphi + b\Phi_{21} - \bar{c}\Phi_{20}\right) \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
N\Psi_1 &= b\left(\text{þ}'\phi_0 + 2\tau\phi_1 - \frac{1}{2}\eth\varphi - a\Phi_{01} + c\Phi_{02}\right) \\
&\quad - \bar{c}\left(\bar{\eth}\phi_0 + 2\rho\phi_1 - \frac{1}{2}\text{þ}\varphi - a\Phi_{00} + c\Phi_{01}\right) \qquad (28') \\
N(\Psi_2 + 2\Lambda) &= c\left(\text{þ}'\phi_0 + 2\tau\phi_1 - \frac{1}{2}\eth\varphi - a\Phi_{01} + c\Phi_{02}\right) \\
&\quad - a\left(\bar{\eth}\phi_0 + 2\rho\phi_1 - \frac{1}{2}\text{þ}\varphi - a\Phi_{00} + c\Phi_{01}\right) \qquad (29')
\end{aligned}
$$

$$
\begin{aligned}
N\Psi_0 &= b(\eth\phi_0 + 2\sigma\phi_1 + b\Phi_{02} - \bar{c}\Phi_{01}) - \bar{c}(\text{þ}\phi_0 + 2\kappa\phi_1 + b\Phi_{01} - \bar{c}\Phi_{00}) \qquad (30) \\
N\Psi_1 &= -a(\text{þ}\phi_0 + 2\kappa\phi_1 + b\Phi_{01} - \bar{c}\Phi_{00}) + c(\eth\phi_0 + 2\sigma\phi_1 + b\Phi_{02} - \bar{c}\Phi_{01}) \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
N\Psi_4 &= c(\text{þ}'\phi_2 - 2\nu\phi_1 - a\Phi_{21} + c\Phi_{22}) - a(\bar{\eth}\phi_2 - 2\lambda\phi_1 - a\Phi_{20} + c\Phi_{21}) \qquad (30') \\
N\Psi_3 &= b(\text{þ}'\phi_2 - 2\nu\phi_1 - a\Phi_{21} + c\Phi_{22}) - \bar{c}(\bar{\eth}\phi_2 - 2\lambda\phi_1 - a\Phi_{20} + c\Phi_{21}) \qquad (31')
\end{aligned}
$$

Note that these equations hold even when $N = 0$. Together with the Maxwell equations (23) they are equivalent to the first integrability conditions we started with. For vacuum and when ξ is a Killing vector, they reduce to those of the FS formalism, provided we assume either that the bivector is nonnull and $\phi_0 = \phi_2 = 0$, $\phi_1 \neq 0$ or that the bivector is null and $\phi_0 = \phi_1 = 0$, $\phi_2 \neq 0$. The latter formalism is indeed a very special case of the present one.

Let us assume that ξ is a KV with a nonsingular bivector and that we have taken a canonical basis for which $\phi_0 = 0 = \phi_2$. Adding Eqs.(29) and (29′) then yields

$$
2\phi_1 \xi \cdot X_2 - b^2\Phi_{22} + 2b\bar{c}\Phi_{21} - c\bar{c}\Phi_{20} - 2ac\Phi_{01} + c^2\Phi_{02} + a^2\Phi_{00} = 0 \qquad (32)
$$

where

$$
X_2 = \mu l - \rho n - \pi m + \tau\bar{m}. \qquad (33)
$$

In vacuum this gives the third vector orthogonal to ξ, as also derived by FS. But we see clearly from Eq.(32) that only under special conditions as the ones described do we get such a third orthogonal vector.

IV. THE NP CONFORMAL KILLING EQUATIONS AND THEIR INTEGRABILITY CONDITIONS

From Eqs.(1)-(4) we see that

$$Ł_{\xi} = \pounds_{\xi} - p\mathfrak{G} - q\overline{\mathfrak{G}} \tag{34}$$

where

$$\mathfrak{G} = \frac{1}{4}\left(\mathcal{P} - \mathcal{P}' + \mathcal{P}^* - \mathcal{P}'^* + 4\xi^{\mu}\zeta_{\mu}.\right) \tag{35}$$

Because of the conformal Killing equations (7), for a (C)KV ξ this reduces to

$$\mathfrak{G} = \frac{1}{2}\left(\mathcal{P} - i\mathcal{S} + \frac{\varphi}{2} + 2\xi^{\mu}\zeta_{\mu}.\right) \tag{36}$$

In terms of the component ϕ_1 of the associated bivector this becomes

$$\mathfrak{G} = \phi_1 + \xi^{\mu}\zeta_{\mu}, \tag{37}$$

i.e.

$$\mathfrak{G} = \phi_1 + \epsilon a + \gamma b - \beta c - \alpha\overline{c}. \tag{38}$$

To fix the gauge requires the specification of two real parameters corresponding to boost and phase. It is therefore possible to have a preferred boost or a preferred phase (or both). The necessary and sufficient condition for the boost-part of the gauge to be preferred is that $Ł_{\xi}\eta = \pounds_{\xi}\eta$ for any scalar quantity η of weight (p,p). Therefore, for a preferred boost we must have

$$\mathfrak{G} + \overline{\mathfrak{G}} = 0. \tag{39}$$

Similarly, the necessary and sufficient condition for the phase-part of the gauge to be preferred is that $Ł_{\xi}\eta = \pounds_{\xi}\eta$ for any scalar quantity η of weight $(p,-p)$, i.e. that

$$\mathfrak{G} - \overline{\mathfrak{G}} = 0. \tag{40}$$

The necessary and sufficient condition for the full gauge to be preferred is that $\mathfrak{G}$ vanish, i.e. that

$$\phi_1 = -\epsilon a - \gamma b + \beta c + \alpha\overline{c}. \tag{41}$$

Recalling that

$$þ = D - p\epsilon - q\overline{\epsilon}, \tag{42}$$

and similarly for its companions, it is straightforward to write Eqs.(17)-(22) in NP notation. They become

$$Db - (\epsilon + \overline{\epsilon})\, b = -\kappa c - \overline{\kappa c} \tag{43}$$

$$Da + (\epsilon + \overline{\epsilon})\, a = \frac{\varphi}{2} - \phi_1 - \overline{\phi_1} + \overline{\pi}c + \pi\overline{c} \tag{44}$$

$$Dc + (\epsilon - \overline{\epsilon})\, c = -\overline{\phi_0} - \overline{\kappa}a + \pi b \tag{45}$$

$$D\phi_1 = \pi\phi_0 - \kappa\phi_2 - \frac{1}{4}D\varphi + b\left(\Lambda - \Phi_{11} - \Psi_2\right) + c\Psi_1 + \overline{c}\Phi_{10} \tag{46}$$

$$D\phi_2 + 2\epsilon\phi_2 = 2\pi\phi_1 - \frac{1}{2}\overline{\delta}\varphi - b\left(\Psi_3 + \Phi_{21}\right) + c\left(\Psi_2 + 2\Lambda\right) + \overline{c}\Phi_{20} \tag{47}$$

$$D\phi_0 - 2\epsilon\phi_0 = -2\kappa\phi_1 - b\left(\Psi_1 + \Phi_{01}\right) + c\Psi_0 + \overline{c}\Phi_{00} \tag{48}$$

together with their companions.

V. AN ILLUSTRATIVE EXAMPLE

Although the commutator approach (Ref.[1,2,3]) seems preferable to a direct solving of the Killing equations and their first integrability conditions we illustrate in this section how the latter approach is facilitated by using preferred tetrads relative to the (C)KVs. To this end we once again[3] find all (C)KVs for the Kimura metric.[9]

The Kimura metric considered by Koutras and Skea[11] is given by

$$ds^2 = \frac{r^2}{b_0}dt^2 - \frac{1}{r^2 b_0^2}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{49}$$

where b_0 is a constant. It is of Petrov type D with a non-zero energy momentum tensor. We can readily construct a tetrad such that follows that

$$\begin{array}{llll} Dt = \frac{\sqrt{b_0}}{r\sqrt{2}}, & Dr = \frac{rb_0}{\sqrt{2}}, & D\theta = 0, & D\phi = 0 \\ \triangle t = \frac{\sqrt{b_0}}{r\sqrt{2}}, & \triangle r = -\frac{rb_0}{\sqrt{2}}, & \triangle\theta = 0, & \triangle\phi = 0 \\ \delta t = 0, & \delta r = 0, & \delta\theta = \frac{1}{r\sqrt{2}}, & \delta\phi = \frac{i}{r\sqrt{2}\sin\theta}. \end{array} \tag{50}$$

The NP spin coefficients are then given by

$$\begin{array}{l} \kappa = \sigma = \lambda = \nu = \tau = \pi = 0 \\ \gamma = \varepsilon = \dfrac{b_0}{2\sqrt{2}}, \qquad \rho = \mu = -\dfrac{b_0}{\sqrt{2}}, \qquad \beta = -\overline{\alpha} = \dfrac{\cot\theta}{2\sqrt{2}r}, \end{array} \tag{51}$$

and

$$\Psi_2 = -\frac{1}{6r^2}, \qquad \Phi_{11} = \frac{1}{4r^2}, \qquad \Lambda = -\frac{b_0^2}{2} + \frac{1}{12r^2} \tag{52}$$

are the only nonzero components of the Riemann tensor.

In view of Eqs.(51) and (52), the conformal Killing equations and their first integrability conditions, Eqs. (43) -(48) and their companions, become

$$\begin{array}{lll} Da = \dfrac{\varphi}{2} - \left(\mathfrak{G} + \overline{\mathfrak{G}}\right) + \dfrac{b_0}{\sqrt{2}}b, & \Delta a = -\dfrac{b_0}{\sqrt{2}}a, & \delta a = \overline{\delta}a = 0 \\ Db = \dfrac{b_0}{\sqrt{2}}b, & \Delta b = \dfrac{\varphi}{2} + \mathfrak{G} + \overline{\mathfrak{G}} - \dfrac{b_0}{\sqrt{2}}a, & \delta b = \overline{\delta}b = 0 \\ Dc = \dfrac{b_0}{\sqrt{2}}c, & \Delta c = -\dfrac{b_0}{\sqrt{2}}c, & \\ \delta c = -\dfrac{\varphi}{2} + \dfrac{b_0}{\sqrt{2}}(a-b) + \mathfrak{G} - \overline{\mathfrak{G}} - \dfrac{\cot\theta}{\sqrt{2}r}\overline{c}, & \overline{\delta}c = \dfrac{\cot\theta}{\sqrt{2}r}c & \end{array} \tag{53}$$

$$\begin{array}{ll} D\phi_0 = \dfrac{b_0}{\sqrt{2}}\phi_0, & \Delta\phi_0 - \dfrac{b_0}{\sqrt{2}}\phi_0 = \dfrac{1}{2}\delta\varphi + \overline{c}b_0^2 \\ \delta\phi_0 = \dfrac{\cot\theta}{\sqrt{2}r}\phi_0, & \overline{\delta}\phi_0 + \dfrac{\cot\theta}{\sqrt{2}r}\phi_0 = \sqrt{2}b_0 + \dfrac{1}{2}D\varphi + bb_0^2 \end{array} \tag{54}$$

$$\begin{array}{ll} D\phi_1 = -\dfrac{1}{4}D\varphi - \dfrac{1}{2}bb_0^2, & \Delta\phi_1 = \dfrac{1}{4}\Delta\varphi + \dfrac{1}{2}ab_0^2 \\ \delta\phi_1 = -\dfrac{b_0}{\sqrt{2}}\phi_0 - \dfrac{1}{4}\delta\varphi + \overline{c}\left(\dfrac{1}{2r^2} - \dfrac{b_0^2}{2}\right), & \overline{\delta}\phi_1 = \dfrac{b_0}{\sqrt{2}}\phi_2 + \dfrac{1}{4}\overline{\delta}\varphi - c\left(\dfrac{1}{2r^2} - \dfrac{b_0^2}{2}\right) \end{array} \tag{55}$$

$$D\phi_2 + \frac{b_0}{\sqrt{2}}\phi_2 = -\frac{1}{2}\overline{\delta}\varphi - cb_0^2, \qquad \Delta\phi_2 = -\frac{b_0}{\sqrt{2}}\phi_2$$
$$\delta\phi_2 + \frac{\cot\theta}{\sqrt{2}r}\phi_2 = -\sqrt{2}b_0\phi_1 - \frac{1}{2}\Delta\varphi - ab_0^2, \qquad \overline{\delta}\phi_2 = \frac{\cot\theta}{\sqrt{2}r}\phi_2. \tag{56}$$

Clearly, the null directions are the principal null directions of the Weyl tensor. These directions are preferred relative to all possible (C)KVs and hence $\mathcal{Q} = 0 = \mathcal{Q}'$. In view of Eqs.(51), we see immediately from Eqs.(12) that

$$\phi_0 = -\frac{b_0}{\sqrt{2}}\overline{c}, \qquad \phi_2 = -\frac{b_0}{\sqrt{2}}c. \tag{57}$$

The gauge is not in any obvious way preferred for all (C)KVs; in fact, in hindsight it will be seen that neither the boost-part nor the gauge part can be chosen in a way that is preferred relative to all six (C)KVs that this metric turns out to possess. Although we could solve the basic equations involved for ϕ_1, it is easier to work in terms of $\mathfrak{G}$, the quantity that is a measure of by how much the given gauge differs from a preferred one for each (C)KV. Alternatively, we could put an arbitrary gauge factor into our spin coefficients, but that too turns out to make the problem more difficult. From Eqs.(38) and (51) we obtain

$$\phi_1 = \mathfrak{G} - \frac{b_0}{2\sqrt{2}}(a+b) + \frac{\cot\theta}{2\sqrt{2}r}(c - \overline{c}). \tag{58}$$

Substituting from Eqs.(57) and (58) into Eqs. (54) and (56) yields only that

$$D\varphi = \sqrt{2}b_0\left(\frac{\varphi}{2} - \mathfrak{G} - \overline{\mathfrak{G}}\right), \quad \Delta\varphi = -\sqrt{2}b_0\left(\frac{\varphi}{2} + \mathfrak{G} + \overline{\mathfrak{G}}\right), \quad \delta\varphi = 0. \tag{59}$$

Subtracting these two equations we get $D\varphi - \Delta\varphi = \sqrt{2}b_0\varphi$. From this we see immediately that the metric cannot have a proper HV.

Substituting Eqs.(57) and (58) into Eqs.(55) and using the Killing equations (53) yields

$$\begin{aligned}
D\mathfrak{G} &= \Delta\mathfrak{G} = 0\\
\delta\mathfrak{G} &= -\frac{\cot\theta}{2\sqrt{2}r}\left(\mathfrak{G} - \overline{\mathfrak{G}} - \frac{\varphi}{2} + \frac{b_0(a-b)}{\sqrt{2}}\right) + \frac{c+\overline{c}}{4r^2\sin^2\theta}\\
\overline{\delta}\mathfrak{G} &= -\frac{\cot\theta}{2\sqrt{2}r}\left(\mathfrak{G} - \overline{\mathfrak{G}} + \frac{\varphi}{2} - \frac{b_0(a-b)}{\sqrt{2}}\right) - \frac{c+\overline{c}}{4r^2\sin^2\theta}
\end{aligned} \tag{60}$$

Eqs.(53), (59) and (60) can now be solved for the unknowns $a, b, c, \varphi, \mathfrak{G}$. We find that

$$\begin{aligned}
a &= r\left(\frac{h_0}{\sqrt{2b_0}} + \frac{l_1}{2\sqrt{2}b_0} + \frac{l_0 t}{\sqrt{2}}\right) - \frac{l_0}{\sqrt{2b_0}}\\
b &= r\left(\frac{h_0}{\sqrt{2b_0}} - \frac{l_1}{2\sqrt{2}b_0} - \frac{l_0 t}{\sqrt{2}}\right) - \frac{l_0}{\sqrt{2b_0}}\\
c &= \frac{-r}{\sqrt{2}}(a_0\cos\phi + b_0\sin\phi) + \frac{ir}{\sqrt{2}}\left[c_0\sin\theta - \cos\theta\,(a_0\sin\phi - b_0\cos\phi)\right]\\
\varphi &= r\,(l_1 + 2l_0b_0t)\\
\mathfrak{G} &= -\frac{l_0\sqrt{b_0}}{2} + \frac{i}{2\sin\theta}(a_0\sin\phi - b_0\cos\phi)
\end{aligned} \tag{61}$$

where $l_0, l_1, h_0, c_0, a_0, b_0$ are integration constants. Putting all but one of these equal to zero in turn and using Eq.(14) we find, in coordinates (t, r, θ, ϕ), the two proper CKVs

$$\begin{aligned} l_0 = 1: \qquad & \xi^\mu_{(1)} = \left(-\frac{1}{r}, r^2 b_0 t, 0, 0\right) \\ l_1 = 1: \qquad & \xi^\mu_{(2)} = \left(0, \frac{r^2}{2}, 0, 0\right) \end{aligned} \tag{62}$$

with respective conformal factors $\varphi = 2rb_0 t$ and $\varphi = r$, as well as the four Killing vectors

$$\begin{aligned} h_0 = 1: \qquad & \xi^\mu_{(3)} = (1, 0, 0, 0) \\ c_0 = 1: \qquad & \xi^\mu_{(4)} = (0, 0, 0, 1) \\ a_0 = 1: \qquad & \xi^\mu_{(5)} = (0, 0, \cos\phi, -\cot\theta \sin\phi) \\ b_0 = 1: \qquad & \xi^\mu_{(6)} = (0, 0, \sin\phi, \cot\theta\cos\phi)\,. \end{aligned} \tag{63}$$

Noting from the last of Eqs.(61) that $\mathfrak{G}+\overline{\mathfrak{G}} = -l_0\sqrt{b_0}$ and $\mathfrak{G}-\overline{\mathfrak{G}} = \frac{i}{\sin\theta}\left(a_0 \sin\phi - b_0 \cos\phi\right)$ we see from Eqs.(62) and (63) that $\mathfrak{G}+\overline{\mathfrak{G}}$ vanishes for all but $\xi_{(1)}$. Therefore, the four KVs and the proper CKV $\xi_{(2)}$ have the boost-part of the given gauge preferred. Similarly, the two proper CKVs and the KVs $\xi_{(3)}$ and $\xi_{(4)}$ have the phase part of the given gauge preferred.

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